NORTH SYDNEY GIRLS HIGH SCHOOL

2006
TRIAL HSC EXAMINATION

Mathematics Extension 2

General Instructions

- Reading Time – 5 minutes
- Working Time – 3 hours
- Write using black or blue pen
- Board approved calculators may be used
- A table of standard integrals is provided at the back of this page
- All necessary working should be shown in every question

Total Marks – 120

Attempt Questions 1–8

All questions are of equal value

NAME:____________________ **TEACHER:** ____________________

NUMBER:____________________

QUESTION	MARK
1	/15
2	/15
3	/15
4	/15
5	/15
6	/15
7	/15
8	/15
TOTAL	/120

Total Marks – 120
Attempt Questions 1–8
All questions are of equal value

Begin each question in a NEW BOOKLET.

Question 1 (15 marks) **Marks**

a) Evaluate

i) $\displaystyle\int_0^{\frac{\pi}{4}} \frac{\cos x}{1+\sin^2 x}\,dx$. **2**

ii) $\displaystyle\int_0^{\sqrt{2}} \sqrt{4-x^2}\,dx$ using the substitution $x = 2\sin\theta$. **3**

b) Find $\displaystyle\int x^2 e^x\,dx$. **2**

c) i) Write $\dfrac{4x^2+11x-8}{(x+2)(x^2-x+1)}$ in the form $\dfrac{A}{x+2}+\dfrac{Bx+C}{x^2-x+1}$ **2**

ii) Hence evaluate $\displaystyle\int_{-1}^{0} \frac{4x^2+11x-8}{(x+2)(x^2-x+1)}\,dx$. **2**

d) i) Prove that $\displaystyle\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$. **2**

ii) Hence find $\displaystyle\int_0^1 x(1-x)^9\,dx$. **2**

Question 2: Begin a new booklet (15 marks) **Marks**

a) If $z = 2 + 3i$ and $w = 1 - i$, find each of the following, expressing your answer in the form $x + iy$:

i) $z + \overline{w}$ **1**

ii) $\dfrac{z-2}{w}$ **1**

b) Write $-\sqrt{3} + i$ in modulus-argument form and hence evaluate $(-\sqrt{3} + i)^{12}$. **2**

c) i) On the Argand diagram, clearly indicate the region containing all points representing the complex number z which satisfies the following conditions:

$$0 \leq \arg [z - (1 + i)] \leq \frac{3\pi}{4}, \quad |z-1| \leq |z-3| \text{ and } \operatorname{Re} z \geq 0$$ **4**

ii) Hence find the range of values for $\arg z$ when z lies in the region shaded in i) above. **1**

d) Let z be a complex number such that $|z| = 1$ and $\arg z = \theta$, where $0 \leq \theta \leq \dfrac{\pi}{2}$.

i) On an Argand diagram, illustrate the points P and Q which represent z and z^2 respectively, clearly indicating their relationship. **2**

ii) Evaluate, in terms of θ,

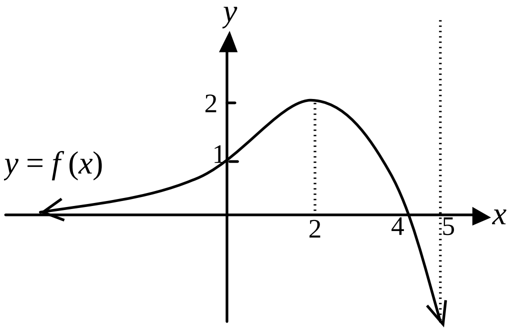
α) $\left|\dfrac{2}{1-z^2}\right|$ **1**

β) $\arg\left(\dfrac{2}{1-z}\right)$ **1**

e) Suppose that w is the complex number $a + ib$. For what conditions on a and b is $w + \dfrac{1}{w}$ purely real? **2**

Question 3: Begin a new booklet (15 marks) **Marks**

a) The graph of $y = f(x)$ is illustrated. The line $y = 0$ is a horizontal asymptote and $x = 5$ is a vertical asymptote.



Using the separate page of graphs provided, sketch each of the graphs below. In each case, clearly label any maxima or minima, intercepts and the equations of any asymptotes.

i) $y = \dfrac{1}{f(x)}$ **2**

ii) $y^2 = f(x)$ **2**

iii) $y = f(|x|)$ **2**

iv) $y = [f(x)]^3$ **2**

v) $y = e^{f(x)}$ **2**

b) Consider the function $f(x) = x - \ln(x^2 + 1)$ for $x \geq 0$.

i) Show that $f'(x) \geq 0$ for $x \geq 0$ **1**

ii) Hence deduce that $x > \ln(x^2 + 1)$ for $x > 0$. **1**

c) Two sides of a triangle are in the ratio 3:1 and the angles opposite these sides differ by $\dfrac{\pi}{6}$. Show that the smaller of the two angles is $\tan^{-1}\left(\dfrac{1}{6-\sqrt{3}}\right)$. **3**

Question 4: Begin a new booklet (15 marks) **Marks**

a) The area enclosed by the curve $xy = 4$ and the line $y = 5 - x$ is rotated about the line $y = 5$ to form a solid.

i) Draw a diagram to illustrate the region. **1**

ii) By taking slices of the solid perpendicular to the axis of rotation, show that the volume of the solid is given by

$$V = \pi\int_1^4 \left\{\left(5 - \frac{4}{x}\right)^2 - x^2\right\}dx$$ **3**

iii) Hence, find the volume of the solid correct to 2 significant figures. **2**

b) i) Prove by Mathematical Induction that if n is a positive integer, then

$$2^{(n+4)} > (n+4)^2.$$ **3**

ii) By choosing a suitable substitution, or otherwise, show that if a is a positive integer, then $2^{3(a+4)} > 9(a+4)^2$. **1**

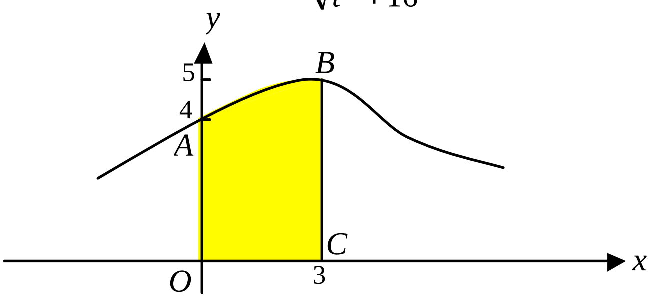
c) i) If $I_n = \int \tan^n x \sec x \, dx$ for integral values of $n \geq 0$, show that

$$nI_n = \tan^{n-1}x \sec x - (n-1)I_{n-2} \text{ for } n \geq 2.$$ **3**

ii) Hence evaluate $\int_0^{\frac{\pi}{4}} \frac{\sin^3 x}{\cos^4 x} dx$. **2**

Question 5: Begin a new booklet (15 marks) **Marks**

a) The diagram shows part of the curve whose parametric equations are given by $x = t + 3$ and $y = \dfrac{20}{\sqrt{t^2 + 16}}$.



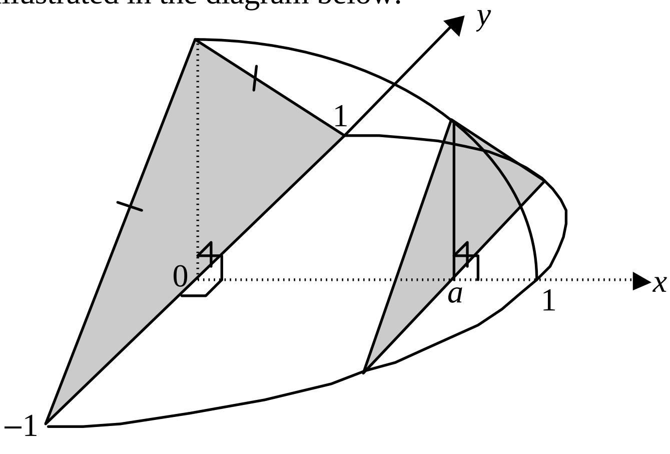
i) Find the values of t that correspond to each of the points A and B on the curve. **1**

ii) A solid is formed by rotating the region $OABC$ about the y-axis. Use the method of cylindrical shells to express the volume in the form

$$V = 40\pi \int_{-3}^{0} \frac{t+3}{\sqrt{t^2+16}}\, dt .$$ **3**

iii) Hence show that the volume of the solid is $40\pi(3\ln 2 - 1)$ unit^3. **3**

b) The base of a solid is the semi-circular region of radius 1 unit in the x-y plane as illustrated in the diagram below.



Each cross-section perpendicular to the x-axis is an isosceles triangle with each of the two equal sides three quarters the length of the third side.

i) Show that the area of the triangular cross-section at $x = a$ is $\frac{\sqrt{5}}{2}(-a^2$. **2**

ii) Hence find the volume of the solid. **3**

c) i) Find the sum of the series $1 + x + x^2 + x^3 + \ldots + x^n$ **1**

ii) Hence find the sum of the series $x + 2x^2 + 3x^3 + \ldots + nx^n$ **2**

Question 6: Begin a new booklet (15 marks) **Marks**

a) The velocity, v ms^{-1}, of a particle moving on the x-axis is given by

$$v^2 = 7 + 20x - 3x^2$$

i) Show that the particle is moving in simple harmonic motion and find the centre of the motion. **2**

ii) Find also the amplitude and period of the motion. **2**

b) A particle P is thrown downwards in a medium where the resistive force is proportional to the speed.

i) Taking the downward direction as positive, explain why $\ddot{x} = g - kv$, where g is the acceleration due to gravity and $k > 0$. **1**

The initial speed is U ms^{-1} and the particle is thrown from a point T which is d metres above a fixed point O, which is taken as the origin.

ii) Show that the velocity, v ms^{-1}, at any time, t seconds, is given by

$$v = \frac{g}{k} - \left(\frac{g - kU}{k}\right)e^{-kt}.$$ **2**

iii) Show that the displacement, x metres, at any time, t seconds, is given by $$x = \frac{gt - kd}{k} + \left(\frac{g - kU}{k^2}\right)\left(e^{-kt} - 1\right)$$ **3**

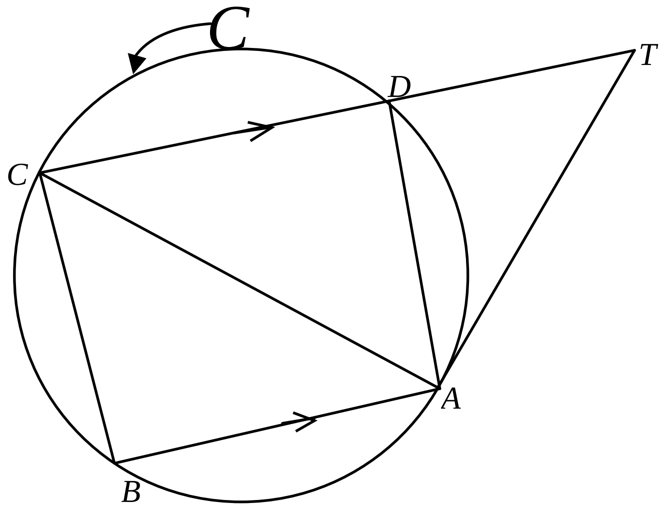
iv) An identical particle Q is dropped from O at the same instant that P is thrown down from T. Use the above results to write down expressions for v and x as functions of t for the particle Q. **2**

v) The particles P and Q collide. Show that the speed at which the particles collide is $\left|U - kd\right|$ ms^{-1}.

(Note: the speed of collision is the difference between the two speeds.) **3**

Question 7: Begin a new booklet (15 marks) **Marks**

a) The points *A*, *B*, *C* and *D* lie on the circle C_1. From the exterior point *T*, a tangent is drawn to point *A* on C_1. The line *CT* passes through *D* and *TC* is parallel to *AB*.



i) Copy or trace the diagram onto your page.

ii) Prove that ΔADT is similar to ΔABC. **3**

The line *BA* is produced through *A* to point *M*, which lies on a second circle C_2. The points *A*, *D*, *T* also lie on C_2 and the line *DM* crosses *AT* at *O*.

iii) Show that ΔOMA is isosceles. **2**

iv) Show that $TM = BC$. **2**

b) i) Prove that the normal to the hyperbola $xy = 4$ at the point $P(2p, \frac{2}{p})$ is given by $p^3x - py = 2(p^4 - 1)$. **2**

ii) If the normal meets the hyperbola again at $Q(2q, \frac{2}{q})$ prove that $p^3q = -1$. **2**

iii) Hence prove that there exists only one chord which is normal to the hyperbola at both ends and find its equation. **4**

Question 8: Begin a new booklet (15 marks) **Marks**

a) i) Find in modulus-argument form, the four roots of -16 and illustrate these on the Argand diagram. **3**

ii) Hence or otherwise, write $z^4 + 16$ as a product of two quadratic factors with real coefficients. **2**

iii) Let α be the root of $z^4 = -16$ which has a principal argument between 0 and $\frac{\pi}{2}$. Show that $\alpha + \frac{\alpha^3}{4} + \frac{\alpha^5}{16} + \frac{\alpha^7}{64} = 0$. **3**

b) If one root of the equation $x^3 - px^2 + qx - r = 0$ is equal to the product of the other two roots, show that $r(p + 1)^2 = (q + r)^2$. **3**

c) If $f(xy) = f(x) + f(y)$ for all $x, y \neq 0$ prove that

i) $f(x^3) = 3f(x)$ **1**

ii) $f(1) = f(-1) = 0$ **2**

iii) $f(x)$ is an even function **1**

End of paper

Question 1:

a) i) $\int_0^{\pi/4} \frac{\cos x}{1+\sin^2 x}\,dx$

$= \left[\tan^{-1}(\sin x)\right]_0^{\pi/4}$

$= \tan^{-1}(\sin \frac{\pi}{4}) - \tan^{-1}(\sin 0)$

$= \tan^{-1}(\frac{1}{\sqrt{2}})$

ii) $\int_0^{\sqrt{2}} \sqrt{4-x^2}\,dx$ let $x = 2\sin\theta$

$\frac{dx}{d\theta} = 2\cos\theta$

$dx = 2\cos\theta\, d\theta$

If $x = 0, \theta = 0$

$x = \sqrt{2}, \theta = \frac{\pi}{4}$

$\therefore \int_0^{\sqrt{2}} \sqrt{4-x^2}\,dx$

$= \int_0^{\pi/4} \sqrt{4-4\sin^2\theta}\,.\,2\cos\theta\,d\theta$

$= \int_0^{\pi/4} 4\cos^2\theta\,d\theta$

$= 2\int_0^{\pi/4} (\cos 2\theta + 1)\,d\theta$

$= 2\left[\frac{1}{2}\sin 2\theta + \theta\right]_0^{\pi/4}$

$= 2\left[\frac{1}{2}\sin\frac{\pi}{2} + \frac{\pi}{4} - (\frac{1}{2}\sin 0 + 0)\right]$

$= 2\left[\frac{1}{2} + \frac{\pi}{4}\right]$

$= 1 + \frac{\pi}{2}$

b) $\int x^2 e^x\,dx$

$= \int x^2 \frac{d(e^x)}{dx}\,.\,dx$

$= e^x x^2 - \int e^x \frac{d(x^2)}{dx}\,dx$

$= e^x x^2 - 2\int x e^x\,dx$

$= e^x x^2 - 2\int x \frac{d(e^x)}{dx}\,dx$

$= e^x x^2 - 2xe^x + 2\int e^x\,dx$

$= x^2 e^x - 2xe^x + 2e^x + C$

c) (i) $\frac{4x^2+11x-8}{(x+2)(x^2-x+1)} = \frac{A}{x+2} + \frac{Bx+C}{x^2-x+1}$

$\therefore 4x^2+11x-8 \equiv A(x^2-x+1) + (Bx+C)(x+2)$

If $x = -2$:

$16-22-8 = A(4+2+1)$

$-14 = 7A$

$A = -2$

If $x = 0$: $-8 = A + 2C$ but $A = -2$

$-8 = -2 + 2C$

$2C = -6$

$C = -3$

If $x = 1$:

$4 + 11 - 8 = A(1-1+1) + (B+C)(3)$

$7 = -2 + 3(B) - 9$

$18 = 3B$

$B = 6$

$\therefore \frac{4x^2+11x-8}{(x+2)(x^2-x+1)} = \frac{-2}{x+2} + \frac{6x-3}{x^2-x+1}$

ii) $\int_{-1}^{0} \frac{4x^2+11x-8}{(x+2)(x^2-x+1)}\,dx$

$= \int_{-1}^{0} \left(\frac{-2}{x+2} + \frac{3(2x-1)}{x^2-x+1}\right)dx$

$= \left[-2\ln(x+2) + 3\ln(x^2-x+1)\right]_{-1}^{0}$

$= -2\ln 2 + 3\ln 1 - (-2\ln 1 + 3\ln 3)$

$= -2\ln 2 - 3\ln 3$

$= -\ln 108$

d) i) $\int_0^a f(x)\,dx$ let $x = a-u$

$dx = -du$

If $x = 0, u = a$

$x = a, u = 0$

$= \int_a^0 f(a-u)\,.\,-du$

$= \int_0^a f(a-u)\,du$

$= \int_0^a f(a-x)\,dx$ as required.

ii) $\int_0^1 x(1-x)^{99}\,dx$

$= \int_0^1 (1-x)x^{99}\,dx$

$= \int_0^1 (x^{99} - x^{100})\,dx$

$= \left[\frac{x^{100}}{100} - \frac{x^{101}}{101}\right]_0^1$

$= \frac{1}{100} - \frac{1}{101}$

$= \frac{1}{10100}$

Question 2:

a) i) $z + w = 2+3i + 1+i$
$= 3+4i$

ii) $\frac{z-2}{w} = \frac{2+3i-2}{1-i}$

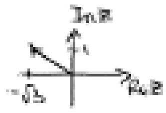
$= \frac{3i(1+i)}{(1-i)(1+i)}$

$= \frac{3i-3}{1+1}$

$= -\frac{3}{2} + \frac{3i}{2}$

b) $|-\sqrt{3}+i| = 2$

$\arg(-\sqrt{3}+i) = \frac{5\pi}{6}$



$\therefore -\sqrt{3}+i = 2\left(\cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6}\right)$

Now $(-\sqrt{3}+i)^{12}$

$= \left[2\left(\cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6}\right)\right]^{12}$

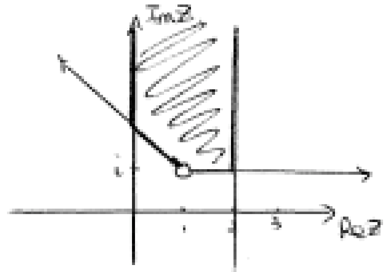
$= 2^{12}\left(\cos\left(\frac{5\pi}{6}\times 12\right) + i\sin\left(\frac{5\pi}{6}\times 12\right)\right)$ by de Moivre

$= 2^{12}(\cos 10\pi + i\sin 10\pi)$

$= 2^{12}(1 + 0i)$

$= 2^{12}$

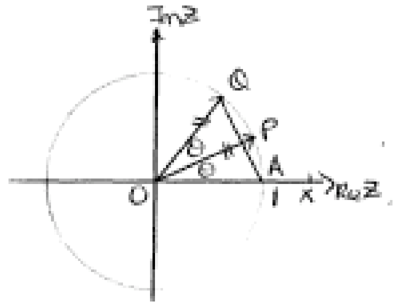
c) i)



↳ Required region

(ii) $\tan^{-1}\left(\frac{1}{2}\right) < \arg z \le \frac{\pi}{2}$

d) i)



ii) α) $\left|\frac{2}{1-z^2}\right| = \frac{|2|}{|1-z^2|}$

Now $1-z^2$ is the vector $\overrightarrow{QA}$

$\therefore |1-z^2|^2 = 1^2 + 1^2 - 2(1)(1)\cos 2\theta$

$= 2 - 2\cos 2\theta$

$= 2 - 2(1 - 2\sin^2\theta)$

$= 4\sin^2\theta$

$\therefore |1-z^2| = 2\sin\theta$ as $0 \le \theta \le \frac{\pi}{2}$

$\therefore \left|\frac{2}{1-z^2}\right| = \frac{2}{2\sin\theta}$

$= \operatorname{cosec}\theta$

β) $\arg\left(\frac{2}{1-z}\right) = \arg 2 - \arg(1-z)$

Now $1-z = \overrightarrow{PA}$

$\arg(1-z) = -\angle PAO$

$= -\left(\frac{\pi-\theta}{2}\right)$

$= \frac{\theta-\pi}{2}$

$\arg 2 = 0$

$\therefore \arg\left(\frac{2}{1-z}\right) = 0 - \left(\frac{\theta-\pi}{2}\right) = \frac{\pi-\theta}{2}$

e) $w + \frac{1}{w} = a+ib + \frac{1}{a+ib}$

$= a+ib + \frac{a-ib}{a^2+b^2}$

For this to be pure real

$\text{Im}\left(w + \frac{1}{w}\right) = 0$

$\therefore\ b - \frac{b}{a^2+b^2} = 0,\ a^2+b^2 \neq 0$

$b(a^2+b^2-1) = 0$

$\therefore\ b = 0 \quad \text{or} \quad a^2+b^2 = 1$
$(a \neq 0)$

Question 3:

a) See separate sheet.

b) i) $f(x) = x - \ln(x^2+1) \qquad x \geq 0$

$f'(x) = 1 - \frac{2x}{x^2+1}$

$= \frac{x^2+1-2x}{x^2+1}$

$= \frac{(x-1)^2}{x^2+1}$

as $(x-1)^2 \geq 0$ if $x \geq 0$ and $x^2+1 > 0$ for all x then $\frac{(x-1)^2}{x^2+1} \geq 0$ for $x \geq 0$

$\therefore\ f'(x) \geq 0$

ii) If $x = 0$ $f(0) = 0 - \ln(0+1)$
$= 0$

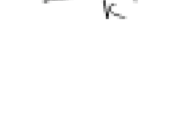
as $f'(x) \geq 0$ for all $x \geq 0$ the function $f(x)$ is increasing or stationary with a minimum value of 0 at $x = 0$

$\therefore\ f(x) \geq 0$ for $x \geq 0$

$\therefore\ x - \ln(x^2+1) \geq 0$ for $x \geq 0$

i.e. $x \geq \ln(x^2+1)$ for $x \geq 0$

c)



let $\alpha < \beta$

$\therefore\ \beta = \alpha + \frac{\pi}{6}$

$\frac{\sin\alpha}{k} = \frac{\sin(\alpha + \frac{\pi}{6})}{3k}$

$3\sin\alpha = \sin(\alpha + \frac{\pi}{6})$

$= \sin\alpha \cdot \frac{\sqrt{3}}{2} + \cos\alpha \cdot \frac{1}{2}$

$6\sin\alpha = \sqrt{3}\sin\alpha + \cos\alpha$

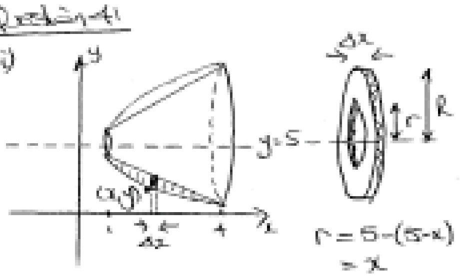
$\sin\alpha\,(6 - \sqrt{3}) = \cos\alpha$

$\tan\alpha = \frac{1}{6-\sqrt{3}}$

$\alpha = \tan^{-1}\left(\frac{1}{6-\sqrt{3}}\right)$ as α is acute

Question 4:

a) i)



$r = 5-(5-x)$
$= x$

$R = 5 - y$
$= 5 - \frac{4}{x}$

(ii) Take a slice Δx thick, perpendicular to $y = 5$ at (x,y) on $xy = 4$. This washer has volume

$\Delta V \doteqdot \pi(R^2 - r^2)\Delta x$

$= \pi\left((5 - \frac{4}{x})^2 - x^2\right)\Delta x$

Vol. of solid is sum of all such slices

$\therefore\ V \doteqdot \sum_{x=1}^{4} \pi\left((5-\frac{4}{x})^2 - x^2\right)\Delta x$

$= \lim_{\Delta x \to 0} \sum_{x=1}^{4} \pi\left((5-\frac{4}{x})^2 - x^2\right)\Delta x$

$= \pi\int_1^4 \left[(5-\frac{4}{x})^2 - x^2\right]dx$ as required

iii) $= \pi\int_1^4 \left(25 - \frac{40}{x} + \frac{16}{x^2} - x^2\right)dx$

$= \pi\left[25x - 40\ln x - \frac{16}{x} - \frac{x^3}{3}\right]_1^4$

$= \pi\left[100 - 40\ln 4 - 4 - \frac{64}{3} - (25 - 16 - \frac{1}{3})\right]$

$= 33.138\ldots$

$= 33$ (2 sig fig)

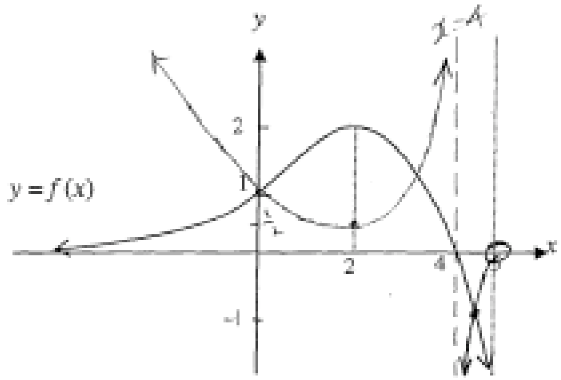
$\therefore$ Volume is 33 cubic units.

4.

Student Number: Answers

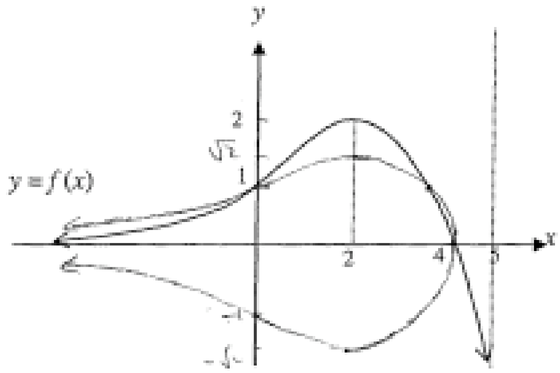
Use this page for your answers to Question 3a)

i)



$y = \frac{1}{f(x)}$

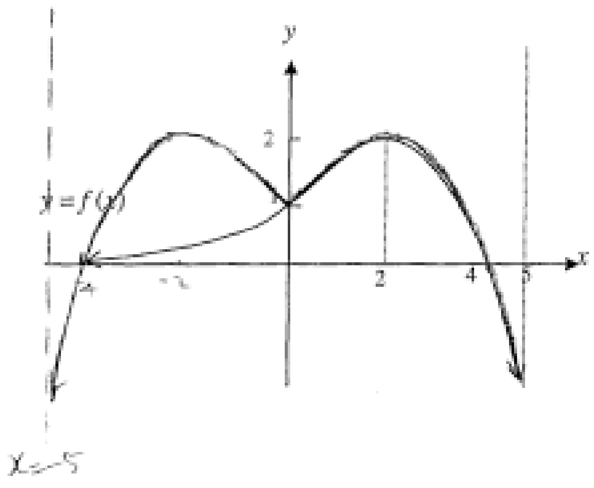
ii)



$y^2 = f(x)$

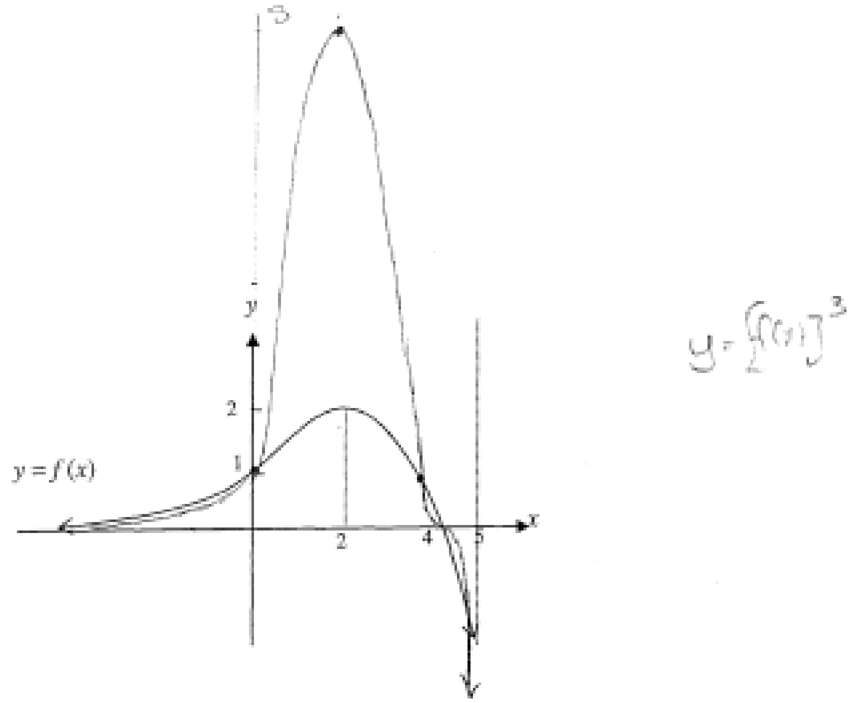
$y = \pm\sqrt{f(x)}$

iii)

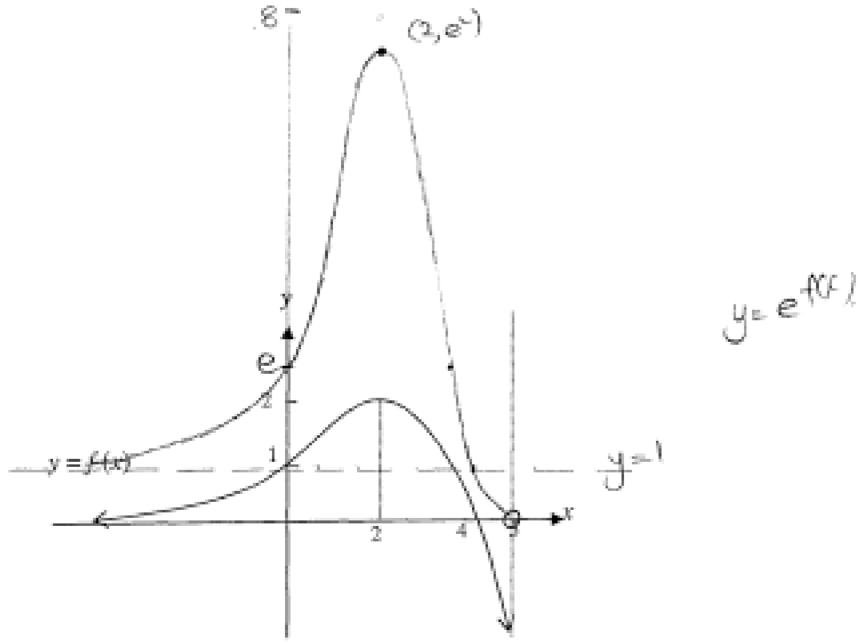


$y = f(|x|)$

iv)



v)



bi) To prove $2^{n+4} > (n+4)^2$ for $n \in J^+$

Test when $n=1$

LHS $= 2^{1+4}$ $= 2^5$ $= 32$ RHS $= (1+4)^2$ $= 5^2$ $= 25$

LHS > RHS ∴ True for $n=1$

Assume true for $n=k$

ie $2^{k+4} > (k+4)^2$ is true

ie $2^{k+4} - (k+4)^2 > 0$

Test if true for $n=k+1$

ie To prove $2^{k+5} > (k+1+4)^2$

or $2^{k+5} > (k+5)^2$

Now $2^{k+5} - (k+5)^2$

$= 2(2^{k+4}) - (k^2+10k+25)$

$= 2(2^{k+4} - k^2 - 8k - 16) + k^2 + 6k + 7$

$= 2(2^{k+4} - (k+4)^2) + k^2 + 6k + 7$

but $2^{k+4} - (k+4)^2 > 0$ by the assumption and $k^2+6k+7>0$ if $k>0$

∴ $2(2^{k+4} - (k+4)^2) + k^2 + 6k + 7 > 0$

∴ $2^{k+5} - (k+5)^2 > 0$

∴ $2^{k+5} > (k+5)^2$

∴ true for $n=k+1$ when assumed true for $n=k$. True for $n=1$ ∴ true for $n=1+1=2$ then $n=2+1=3$ and so on for all $n \in J^+$

ii) let $n+4 = 3(a+4)$

$= 3a+12$

∴ $n = 3a+8$

substituting into $2^{n+4} > (n+4)^2$

gives $2^{3(a+4)} > (3a+8+4)^2$

$= (3(a+4))^2$

$= 9(a+4)^2$

∴ $2^{3(a+4)} > 9(a+4)^2$

as required.

c) i) $I_n = \int \tan^n x \sec x \, dx \qquad n \geq 0$

$= \int \tan^{n-1} x \, \frac{d}{dx}(\sec x) \, dx$

∴ $I_n = \sec x \tan^{n-1} x - \int \sec x (n-1) \tan^{n-2} x \cdot \sec^2 x \, dx \qquad n \geq 2$

$= \sec x \tan^{n-1} x - (n-1) \int \sec x \tan^{n-2} x (1+\tan^2 x) \, dx$

$= \sec x \tan^{n-1} x - (n-1) [I_n + I_{n-2}]$

∴ $(n-1) I_n + I_n = \sec x \tan^{n-1} x - (n-1) I_{n-2}$

$n I_n = \sec x \tan^{n-1} x - (n-1) I_{n-2}$

$I_n = \frac{1}{n} \sec x \tan^{n-1} x - \frac{n-1}{n} I_{n-2} \qquad n \geq 2$

ii) $\int_0^{\pi/4} \frac{\sin^3 x}{\cos^4 x} \, dx$

$= \int_0^{\pi/4} \tan^3 x \sec x \, dx$

$= \left[\frac{1}{3} \sec x \tan^2 x\right]_0^{\pi/4} - \frac{2}{3} \int_0^{\pi/4} \sec x \tan x \, dx$

$= \frac{1}{3} \sec \frac{\pi}{4} \cdot 1^2 - 0 - \frac{2}{3} \left[\sec x\right]_0^{\pi/4}$

$= \frac{1}{3} \cdot \sqrt{2} - \frac{2}{3} [\sqrt{2} - 1]$

$= \frac{2}{3} - \frac{\sqrt{2}}{3}$

$= \frac{1}{3}(2 - \sqrt{2})$

Question 5

a) i) $A = (0,4)$

∴ $0 = t+3$ and $4 = \frac{20}{\sqrt{t^2+16}}$ is true for $t=-3$

∴ $t = -3$ at A

$B = (3,5)$

∴ $3 = t+3$ and $5 = \frac{20}{\sqrt{t^2+16}}$ is true for $t=0$

∴ $t = 0$ at B.

ii) Take a slice of the area at (x,y) on the curve, with width Δx.

This has volume

$\Delta V \doteqdot 2\pi r h \Delta x$

$= 2\pi x y \Delta x$

Vol of solid is the sum of all such shells

$\therefore V \doteq \sum_{t=-3}^{0} 2\pi x y \Delta x$

$= \lim_{\Delta x \to 0} \sum_{t=-3}^{0} 2\pi x y \Delta x$

but as $\Delta x \to 0$, $\Delta t \to 0$

$\therefore V = \lim_{\Delta t \to 0} \sum_{t=-3}^{0} 2\pi x y \Delta t$

$= \int_{-3}^{0} 2\pi (t+3) \cdot \frac{20}{\sqrt{t^2+16}}\, dt$

$= 40\pi \int_{-3}^{0} \frac{t+3}{\sqrt{t^2+16}}\, dt$ as required

iii) $V = 40\pi \int_{-3}^{0} \left(\frac{t}{\sqrt{t^2+16}} + \frac{3}{\sqrt{t^2+16}} \right) dt$

$= 40\pi \left[(t^2+16)^{\frac{1}{2}} + 3\ln\left(t + \sqrt{t^2+16}\right) \right]_{-3}^{0}$

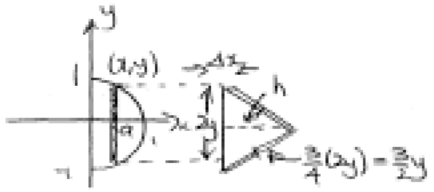
$= 40\pi \left[16^{\frac{1}{2}} + 3\ln\sqrt{16} - 25^{\frac{1}{2}} - 3\ln(-3+5) \right]$

$= 40\pi \left[3\ln 4 - 3\ln 2 - 1 \right]$

$= 40\pi \left(3\ln 2 - 1 \right)$

$\therefore$ Vol of solid is $40\pi(3\ln 2 - 1)$ unit^3.

b)(i)



At $x = a$, $y = \sqrt{1-a^2}$

Area of cross-section

$= \frac{1}{2}(2y)h$ but $h^2 = \left(\frac{3}{2}y\right)^2 - y^2$

$= \frac{5}{4}y$

$h = \frac{\sqrt{5}y}{2}$, $h > 0$

$= y \cdot \frac{\sqrt{5}}{2} y$

$= \frac{\sqrt{5}}{2}\left(\sqrt{1-a^2}\right)^2$

$= \frac{\sqrt{5}}{2}(1-a^2)$ as required

ii) Vol. of slice is

$\Delta V \doteq \frac{\sqrt{5}}{2}(1-x^2)\Delta x$

$\therefore$ Vol. of solid is

$V = \sum_{x=0}^{1} \frac{\sqrt{5}}{2}(1-x^2)\Delta x$

$= \lim_{\Delta x \to 0} \sum_{x=0}^{1} \frac{\sqrt{5}}{2}(1-x^2)\Delta x$

$= \frac{\sqrt{5}}{2}\int_0^1 (1-x^2)\,dx$

$= \frac{\sqrt{5}}{2}\left[x - \frac{x^3}{3} \right]_0^1$

$= \frac{\sqrt{5}}{2}\left[1 - \frac{1}{3} - 0 \right]$

$= \frac{\sqrt{5}}{3}$

$\therefore$ Vol. of solid is $\frac{\sqrt{5}}{3}$ unit^3.

c) i) $1 + x + x^2 + x^3 + \ldots + x^n$

$= \frac{a(1-r^N)}{1-r}$ where $a = 1$, $r = x$, $N = n+1$

$= \frac{1(1-x^{n+1})}{1-x}$

$= \frac{1-x^{n+1}}{1-x}$

ii) $1 + x + x^2 + \ldots + x^n = \frac{1-x^{n+1}}{1-x}$

Differentiating both sides w.r.t x

$1 + 2x + 3x^2 + \ldots + nx^{n-1}$

$= \frac{(1-x)\left[-(n+1)x^n\right] - (1-x^{n+1})(-1)}{(1-x)^2}$

$= \frac{-(n+1)x^n + (n+1)x^{n+1} + 1 - x^{n+1}}{(1-x)^2}$

$= \frac{1 - nx^n - x^n + nx^{n+1}}{(1-x)^2}$

multiplying both sides by x

$x + 2x^2 + 3x^3 + \ldots + nx^n$

$= \frac{x - nx^{n+1} - x^{n+1} + nx^{n+2}}{(1-x)^2}$

$= \frac{-x^{n+1}(1+n) + x(1+nx^{n+1})}{(1-x)^2}$

Question 6:

a) $v^2 = 7 + 20x - 3x^2$

i) $\frac{d(\frac{1}{2}v^2)}{dx} = \frac{d}{dx}\left(\frac{1}{2}(7+20x-3x^2)\right)$

$= \frac{1}{2}(20-6x)$

$= 10-3x$

$= -3(x-\frac{10}{3})$

this is of the form

$\ddot{x} = -n^2(x-b)$

$\therefore$ the particle exhibits SHM about the point where $x = \frac{10}{3}$

ii) Period $= \frac{2\pi}{n}$ but $n = \sqrt{3}$

$\therefore$ period $= \frac{2\pi}{\sqrt{3}} \cdot \frac{\sqrt{3}}{\sqrt{3}}$ sec

$= \frac{2\sqrt{3}\pi}{3}$ sec

The particle stops if $v = 0$

ie $7+20x-3x^2=0$

$3x^2-20x-7=0$

$(3x+1)(x-7)=0$

$x = -\frac{1}{3}, 7$

$\therefore 2a = \frac{1}{3}+7$

$a = \frac{11}{3}$

ie the amplitude is $\frac{11}{3}$ m

b) i) $x=0$, $+\downarrow$, $\downarrow mg$, $\uparrow R = mkv$, $k>0$

Resultant force acting on particle is

$F = mg - mkv$

$\therefore m\ddot{x} = mg - mkv$

$\ddot{x} = g - kv$ where $k>0$

ii) $t=0$, $v=u$, $x=-d$

$x=-d$ T, $+\downarrow$, $x=0$ O

then

$\frac{dv}{dt} = g-kv$

$\frac{dt}{dv} = \frac{1}{g-kv}$

$t = \int \frac{1}{g-kv}\,dv$

$= -\frac{1}{k}\ln(g-kv)+c$

If $t=0$, $v=u$

$\therefore 0 = -\frac{1}{k}\ln(g-ku)+C$

$C = \frac{1}{k}\ln(g-ku)$

$\therefore t = -\frac{1}{k}\ln(g-kv)+\frac{1}{k}\ln(g-ku)$

$-kt = \ln\left(\frac{g-kv}{g-ku}\right)$

$(g-ku)e^{-kt} = g-kv$

$v = \frac{g}{k} - \left(\frac{g-ku}{k}\right)e^{-kt}$

as required.

iii) $\therefore \frac{dx}{dt} = \frac{g}{k} - \left(\frac{g-ku}{k}\right)e^{-kt}$

$x = \frac{gt}{k} - \frac{g-ku}{k(-k)}e^{-kt} + C_2$

If $t=0$, $x=-d$

$\therefore C_2 = -d + \frac{g-ku}{(-k^2)}e^0$

$= -d + \frac{ku-g}{k^2}$

$\therefore x = \frac{gt}{k} + \frac{g-ku}{k^2}e^{-kt} - d + \frac{ku-g}{k^2}$

$x = \frac{gt-kd}{k} + \left(\frac{g-ku}{k^2}\right)(e^{-kt}-1)$

as required

iv) $x=0$, $+\downarrow$, $v=0$

for Q: $v = \frac{g}{k} - \frac{g}{k}e^{-kt}$

$x = \frac{gt}{k} + \frac{g}{k^2}(e^{-kt}-1)$

v) Particles collide if x-values are equal

$\therefore \frac{gt-kd}{k} + \frac{g-kU}{k^2}(e^{-kt}-1) = \frac{gt}{k} + \frac{g}{k^2}(e^{-kt}-1)$

$d + \frac{kU}{k^2}(e^{-kt}-1) = 0$

$\frac{U}{k}(e^{-kt}-1) = -d$

$e^{-kt}-1 = \frac{-dk}{U}$

$e^{-kt} = 1 - \frac{dk}{U}$

$-kt = \ln\left(1-\frac{dk}{U}\right)$

$t = -\frac{1}{k}\ln\left(1-\frac{dk}{U}\right)$

$= \frac{1}{k}\ln\left(\frac{U}{U-kd}\right)$

At that time

P travels at a velocity of v_1 where

$v_1 = \frac{g}{k} - \left(\frac{g-kU}{k}\right)e^{-k\left(\frac{1}{k}\ln\left(\frac{U}{U-kd}\right)\right)}$

$= \frac{g}{k} - \left(\frac{g-kU}{k}\right)\left(\frac{U-kd}{U}\right)$

Q travels at a velocity of v_2 where

$v_2 = \frac{g}{k} - \frac{g}{k}\left(\frac{U-kd}{U}\right)$

speed of collision is

$|v_1 - v_2|$

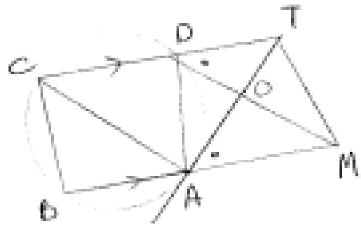
$= \left|-\frac{kU}{k}\left(\frac{U-kd}{U}\right)\right|$

$= |U-kd|$

$\therefore$ speed of collision is $|U-kd|\ ms^{-1}$

Question 7:

a) i)



ii) In $\triangle ADT$ & $\triangle ABC$

1. $\angle TAD = \angle DCA$ ($\angle$ in alternate segment on AD)

$= \angle CAB$ (alternate $\angle$'s; CD$\parallel$BA)

2. $\angle TDA = \angle CBA$ (ext $\angle$ of cyclic quad ABCD)

$\therefore \triangle ADT \parallel\!\parallel \triangle ABC$ (equiangular)

iii) ADTM is a cyclic quad (all on C_2)

$\angle TAM = \angle TDM$ ($\angle$'s on same segment TM in C_2)

$\angle TDM = \angle DMA$ (alternate $\angle$'s; DT$\parallel$AM)

$\therefore \angle TAM = \angle DMA$

$\therefore OA = OM$ (opposite equal angles in $\triangle OMA$)

$\therefore \triangle OMA$ is isosceles.

iv) As $\angle TDM = \angle DMA$ (proven above)

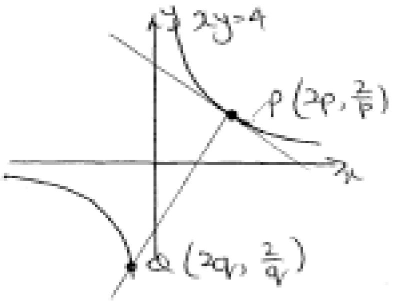
TM = DA ($\angle$'s at circumference are equal in C_2)

As $\angle DCA = \angle CAB$ (proven in ii))

DA = CB ($\angle$'s at circumference are equal in C_1)

$\therefore$ TM = CB as required.

b)



(i) $xy=4$

$P=\left(2p,\frac{2}{p}\right)$

Differentiating w.r.t. x

$x\frac{dy}{dx}+y=0$

$x\frac{dy}{dx}=-y$

$\frac{dy}{dx}=\frac{-y}{x}$

at P: $\frac{dy}{dx}=\frac{-\frac{2}{p}}{2p}$

$=\frac{-1}{p^2}$

$\therefore$ grad. of normal at $P=p^2$

eqn. of normal is

$y-\frac{2}{p}=p^2(x-2p)$

$py-2=p^3x-2p^4$

$p^3x-py=2(p^4-1)$ as required.

ii) $Q\left(2q,\frac{2}{q}\right)$ satisfies this equation

$\therefore\ p^3(2q)-p\left(\frac{2}{q}\right)=2(p^4-1)$

$2p^3q^2-2p=2p^4q-2q$

$p^3q^2-p=p^4q-q$

$p^3q(q-p)=p-q$

$p^3q=-1$ as $p\neq q$

P & Q distinct.

iii) Let PQ be a chord which is normal to the hyperbola at both ends

then $p^3x-py=2(p^4-1)$ and

$q^3x-qy=2(q^4-1)$

also then

$p^3q=-1$ and ①

$q^3p=-1$ ②

from ① $q=\frac{-1}{p^3}$

sub in ② $\left(\frac{-1}{p^3}\right)^3p=-1$

$\frac{-1}{p^8}=-1$, $p\neq0$

$p^8=1$

$p=\pm1$

If $p=1$, $q=-1$ and the chord is $x-y=0$

If $p=-1$, $q=1$ and the chord is $-x+y=0$

$x-y=0$

$\Rightarrow$ there is only one such chord and it is $x-y=0$ ie $y=x$.

Question 8:

a) i) let $z^4=-16$

and $z=r(\cos\theta+i\sin\theta)$

then $z^4=r^4(\cos4\theta+i\sin4\theta)$ by de Moivre.

but $|z^4|=|z|^4$

$=|-16|$

$=16$

$\therefore\ |z|=2=r$

Also

$r^4(\cos4\theta+i\sin4\theta)=-16$

$\therefore 16(\cos4\theta+i\sin4\theta)=-16$

$\cos4\theta+i\sin4\theta=-1$

equating reals/imaginaries

$\cos4\theta=-1$ & $\sin4\theta=0$

$\therefore 4\theta=\pi, 3\pi, 5\pi, 7\pi$ (we need 4 roots)

$\therefore\ \theta=\frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$

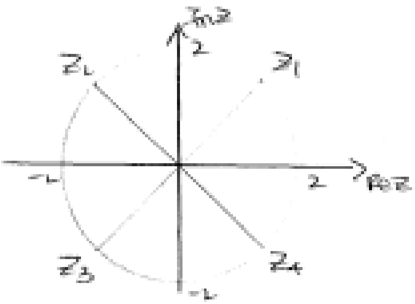
$\therefore$ the 4 roots of -16 are

$z_1 = 2\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$

$z_2 = 2\left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right)$

$z_3 = 2\left(\cos\frac{5\pi}{4} + i\sin\frac{5\pi}{4}\right)$

$z_4 = 2\left(\cos\frac{7\pi}{4} + i\sin\frac{7\pi}{4}\right)$



(ii) Now $z_1 = \overline{z_4}$

$\therefore \quad z_1 z_4 = 2\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right) \cdot 2\left(\cos\frac{\pi}{4} - i\sin\frac{\pi}{4}\right)$

$= 4\left(\cos^2\frac{\pi}{4} + \sin^2\frac{\pi}{4}\right)$

$= 4$

$z_1 + z_4 = 2\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right) + 2\left(\cos\frac{\pi}{4} - i\sin\frac{\pi}{4}\right)$

$= 4\cos\frac{\pi}{4}$

$= \frac{4}{\sqrt{2}}$

$= 2\sqrt{2}$

similarly $z_2 = \overline{z_3}$

$z_2 z_3 = 4$

$z_2 + z_3 = 4\cos\frac{3\pi}{4}$

$= -2\sqrt{2}$

$\therefore z^4 + 16$

$= (z - z_1)(z - z_2)(z - z_3)(z - z_4)$

$= \left(z^2 - (z_1 + z_4)z + z_1 z_4\right)\left(z^2 - (z_2 + z_3)z + z_2 z_3\right)$

$= \left(z^2 - 2\sqrt{2}z + 4\right)\left(z^2 + 2\sqrt{2}z + 4\right)$

(iii) Now $\alpha = 2\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$

by de Moivre:

$\alpha^3 = 8\left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right)$

$= 8\left(-\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$

$\alpha^5 = 32\left(\cos\frac{5\pi}{4} + i\sin\frac{5\pi}{4}\right)$

$= 32\left(-\cos\frac{\pi}{4} - i\sin\frac{\pi}{4}\right)$

$\alpha^7 = 128\left(\cos\frac{7\pi}{4} + i\sin\frac{7\pi}{4}\right)$

$= 128\left(\cos\frac{\pi}{4} - i\sin\frac{\pi}{4}\right)$

$\therefore \alpha + \frac{\alpha^3}{4} + \frac{\alpha^5}{16} + \frac{\alpha^7}{64}$

$= 2\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$

$+ 2\left(-\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$

$+ 2\left(-\cos\frac{\pi}{4} - i\sin\frac{\pi}{4}\right)$

$+ 2\left(\cos\frac{\pi}{4} - i\sin\frac{\pi}{4}\right)$

$= 2(0)$

$= 0$ as required.

(b) let the roots be α, β, γ and $\gamma = \alpha\beta$

then $\alpha + \beta + \gamma = p$ (1)

$\alpha\beta + \alpha\gamma + \beta\gamma = q$ (2)

$\alpha\beta\gamma = r$ (3)

sub $\gamma = \alpha\beta$ into (3):

$\gamma^2 = r$

$\gamma = \pm\sqrt{r}$

but γ is a root of $x^3 - px^2 + qx - r = 0$

$\therefore$ if $\gamma = \sqrt{r}$

$r\sqrt{r} - pr + q\sqrt{r} - r = 0$

$(r+q)\sqrt{r} = r(p+1)$

$(r+q)^2 r = r^2(p+1)^2$

$(r+q)^2 = r(p+1)^2$

similarly if $\gamma = -\sqrt{r}$

$-r\sqrt{r} - pr - q\sqrt{r} - r = 0$

$-r(p+1) = \sqrt{r}(r+q)$

$r(p+1)^2 = (r+q)^2$

$\therefore (q+r)^2 = r(p+1)^2$ as required

c) $f(xy) = f(x) + f(y)$ $\quad xy \neq 0$

i) $f(x^3) = f(x x^2)$

$= f(x) + f(x^2)$

$= f(x) + f(x) + f(x)$

$= 3f(x)$ as required

ii) $f(1) = f(1 \times 1)$

$= f(1) + f(1)$

$\therefore f(1) = 0$

but $f(1) = f(-1 \times -1)$

$= f(-1) + f(-1)$

$\therefore 0 = 2f(-1)$ as $f(1) = 0$

$\therefore f(-1) = 0$

ie $f(1) = f(-1) = 0$ as required.

iii) $f(-x) = f(-1 \times x)$

$= f(-1) + f(x)$

$= 0 + f(x)$ as $f(-1) = 0$

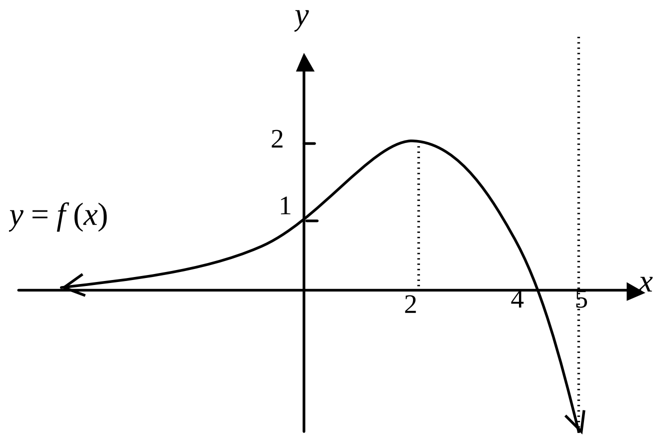
$= f(x)$

$\therefore$ the function is even.

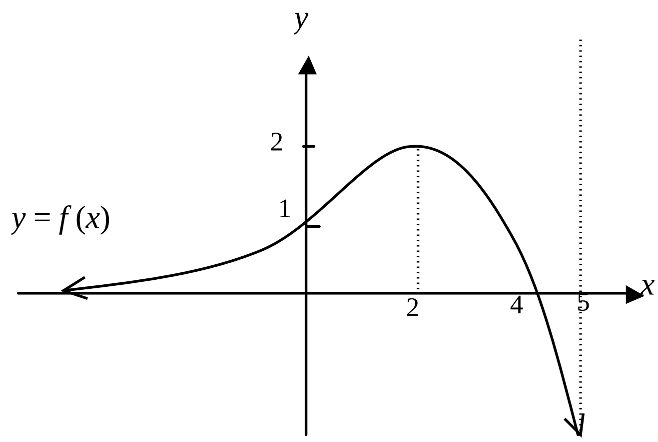
Student Number:_____________________________

Use this page for your answers to Question 3a)

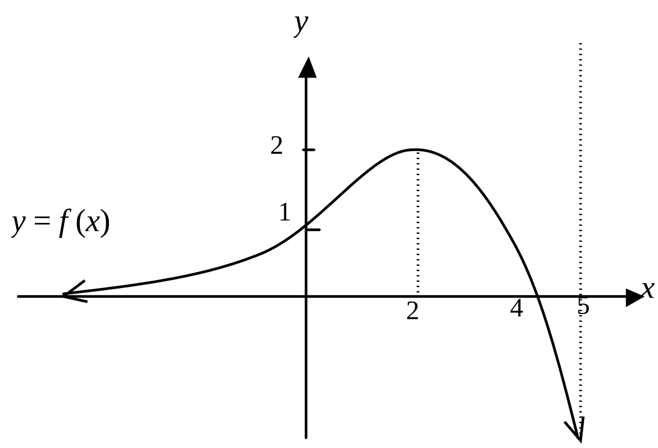
i)



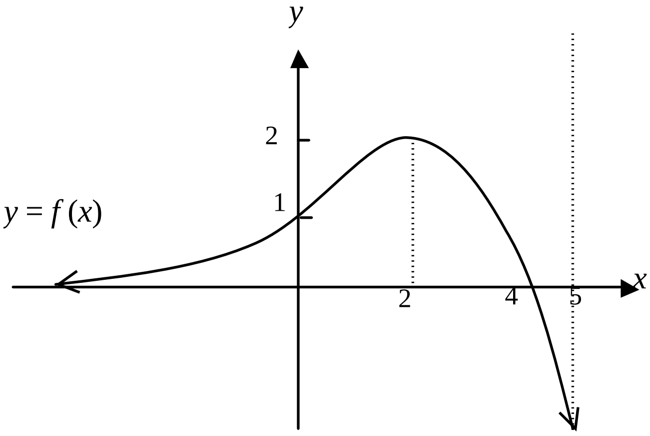
ii)



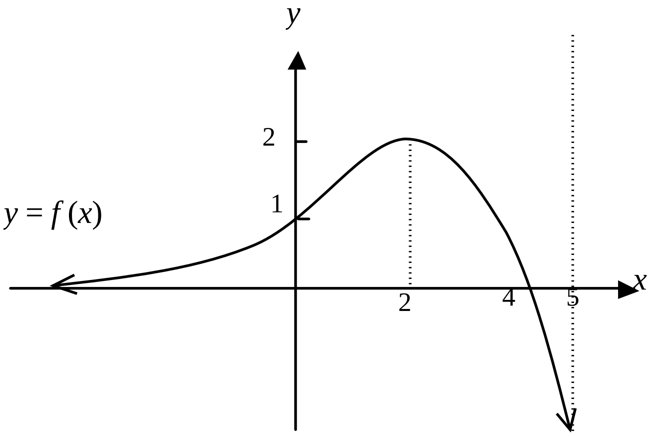
iii)



iv)



v)



Insert this page in your booklet for Question 3